Renormalisability of noncommutative GUT inspired field theories with anomaly safe groups

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ABSTRACT: We consider noncommutative GUT inspired field theories formulated within the enveloping-algebra formalism for anomaly safe compact simple gauge groups. Our theories have only gauge fields and fermions, and we compute the UV divergent part of the one-loop background-field effective action involving two fermionic fields at first order in the noncommutativity parameter θ. We show that, if the second-degree Casimir has the same value for all the irreducible group representations furnished by the fermionic multiplets of the model, then, that UV divergent part can be renormalised by carrying out multiplicative renormalisations of the coupling constant, θ and the fields, along with the inclusion of θ-dependent counterterms which vanish upon imposing the equations of motion. These θ-dependent counterterms have no physical effect since they vanish on-shell. This result along with the vanishing of the UV divergent part of the fermionic four-point functions leads to the unexpected conclusion that the one-loop matter sector of the background-field effective action of these theories is one-loop multiplicatively renormalisable on-shell. We also show that the background-field effective action of the gauge sector of the theories considered here receives no θ-dependent UV divergent contributions at one-loop. We thus conclude that these theories are on-shell one-loop multiplicatively renormalisable at first order in θ.

Contents

1 Introduction

Noncommutative gauge theories with simple groups can only be formulated with the so called enveloping algebra approach, which makes use of Seiberg-Witten maps to relate noncommutative gauge orbits to ordinary ones [1]. Since the Seiberg-Witten maps are generically obtained perturbatively in the noncommutativity parameters θ, the resulting theories, which are invariant under ordinary gauge transformations, involve interaction terms at all orders in θ. This, and the fact that θ has negative mass dimensions, seems to suggest that the theories are only meaningful as effective theories. However, some intriguing results seem to point towards a perturbative self-consistency of the theories: it could well be that for some models the structure imposed by the Seiberg-Witten maps survives quantum corrections, so that the divergences can be absorbed by both multiplicative renormalisations and by physically irrelevant counterterms (e.g., couterterms which vanish on-shell). The first one of these results concerns the fact that the gauge anomaly cancellation conditions have been shown to be, to all orders in θ, equal to their commutative counterparts [2]; this allowed to formulate noncommutative extensions of the Standard Model [3], and GUT theories [4]. Other results concern the renormalisability of the gauge sector at one-loop, observed for a variety of models independently of the matter content [5–12]; in fact, the matter determinants contributing to the one-loop gauge effective action are known to yield renormalisable contributions to all orders in θ , at least for non-chiral theories [13].

Despite these auspicious results, the matter sector –in the fundamental representation– of the theories studied so far –having U(1) and SU(2) as gauge groups– is nonrenormalisable [5, 7, 11] and the lack of renormalisability can be traced back to problematic divergences in four point functions of the matter fields . There are, however, promising exceptions: on the one hand, supersymmetric (S)U(N) theories with adjoint Majorana fermions in a vector multiplet have been shown to be one-loop renormalisable [13], and, on the other, noncommutative GUT inspired theories with arbitrary groups and representations have been shown to be free of the unwelcomed four fermion divergences just mentioned [14, 15].

In this paper we continue the study of the renormalisability of noncommutative GUT inspired theories with no scalar fields, by computing the UV divergent part of the effective action involving two fermion fields. Here, we do it for theories with anomaly safe compact simple gauge groups –groups for which the anomaly coefficient vanishes in all representations– since among these groups one finds the phenomenologically promising SO(10) and E_6. These models have, as a consequence of the anomaly cancellation condition, no vertices of first order in θ in the bosonic part of the classical action, and hence are not sensitive to ambiguities in the trace over bosonic fields [4]. We use the background field method in the Feynman-background-field gauge in conjunction with dimensional regularisation to reconstruct, at first order in θ, the full one-loop UV divergent contribution involving two fermion fields. We do so by using gauge invariance and working out the pole part of the two- and three-point Green functions involving, respectively, two fermion fields and one gauge field and two fermion fields. The result is the following: whenever all the irreducible representations carried by the fermion multiplets of the theory share the same second-degree Casimir, the UV divergences can be renormalised by using multiplicative renormalisation of the coupling constant, the noncommutative matrix parameter $\theta^{\mu\nu}$ and the fields, and by adding θ-dependent counterterms which vanish on-shell, i.e., upon imposing the equation of motion. These θ-dependent counterterms which vanish on-shell have, of course, no physical effect. If one combines this result with the absence of UV divergent contributions to the fermionic four-point function of these theories –see ref. [15]–, one concludes that the one-loop matter sector of the theory is renormalisable on-shell at first order in θ; this is the first time that such property is shown to hold in a noncommutative theory with nonmajorana fermions. The requirement of a common second-degree Casimir for the matter representations can be fulfilled by using a single irreducible representation –as is commonly done in ordinary GUTs such as SO(10) and E_6–, though our renormalisability result is valid for any choice of representation– or combining a representation with its conjugate. Finally, once the matter sector has been seen to be renormalisable, we show by using formal arguments that there are no UV divergent contributions to the gauge sector which are of first order in θ. We thus put forward, for the first time in the literature, a huge family of noncommutative theories with chiral fermionic matter and GUT gauge groups which are one-loop renormalisable at first order in θ, in the physical sense that only the ordinary renormalisation of the coupling constant and a new multiplicative renormalisation of the noncommutative matrix parameter $\theta^{\mu\nu}$ are needed to workout UV finite S-matrix elements: the counterterms –in particular, the a priori problematic θ-dependent counterterms– which are not given by the renormalisations of θ and the coupling constant vanish on-shell. Recall that the free pa-

rameters of our classical noncommutative field theories are the coupling constant and $\theta^{\mu\nu}$.

Now, to make sure that the Hamiltonian formulation of our theories is the elementary one –only one canonical momenta per generalised coordinate–, we shall choose a noncommutative matrix parameter $\theta^{\mu\nu}$ such that $\theta^{0i} = 0$, $i = 1, 2, 3$. Hence, without lose of generality one can say that $\theta^{\mu\nu}$ is characterized by a single noncommutative parameter, say, θ.

The paper is organised as follows. The theory is defined in section 2, where the computation by means of the background field method is also outlined. Section 3 includes the results of the computations of the UV divergent part of the effective action involving two fermion fields, whose renormalisability is discussed in section 4. Section 5 is dedicated to argue in favour of the renormalisability of the gauge sector. Conclusions are presented in section 6. We also include two appendices: appendix A provides the results for the divergent contributions to the Feynman diagrams involved in the computations of section 3, while appendix B gives the results for the beta functions of the physical parameters of the theory, g and θ.

2 The theory and their background field method quantisation

We shall consider a general four-dimensional noncommutative GUT inspired theory with an arbitrary anomaly safe [2] compact simple gauge group and no scalar fields as formulated in ref. [4]. We thus define the theory by means of a noncommutative left-handed chiral multiplet Ψ in an arbitrary representation ρ_Ψ of the gauge group, and an enveloping-algebra valued gauge field A_μ with action

$$S = \int d^4x \; -\frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu} \star F^{\mu\nu} + \bar{\Psi}_L i\slashed{D}\Psi_L, \tag{2.1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star, \qquad D_\mu\psi_L = \partial_\mu\Psi_L - i\rho_\Psi(A_\mu)\star\Psi_L,$$

where, at first order in θ, the noncommutative fields are defined in terms of the ordinary ones a_μ, ψ by the following standard Seiberg-Witten maps,

$$A_\mu = a_\mu + \frac{1}{4}\theta^{\alpha\beta}\{\partial_\alpha a_\mu + f_{\alpha\mu}, a_\beta\} + O(\theta^2),$$

$$\Psi_L = \psi_L - \frac{1}{2}\theta^{\alpha\beta}\rho_\psi(a_\alpha)\partial_\beta\psi_L + \frac{i}{4}\theta^{\alpha\beta}\rho_\psi(a_\alpha)\rho_\psi(a_\beta)\psi_L + O(\theta^2). \tag{2.2}$$

Note that ρ_ψ denotes an arbitrary unitary representation, which can be expressed as a direct sum of irreducible representations, $\rho_\psi = \bigoplus_{r=1}^F \rho_\psi^r$. Accordingly, the fermion fields can be expressed as a direct sum of irreducible multiplets, $\Psi_L = \bigoplus_{r=1}^F \Psi_L^r$, $\psi_L = \bigoplus_{r=1}^F \psi_L^r$.

Upon substituting eq. (2.2) in eq. (2.1) and, then, expanding up to first order in θ, one obtains a classical action for the ordinary fields a_μ^a and ψ_L. Within the enveloping-algebra formalism, the quantisation of the theory defined by this classical action defines the corresponding noncommutative field theory at first order in θ. It has been shown in ref. [4] that for compact simple gauge groups the anomaly cancellation condition [2] makes the first order in θ contribution coming from the noncommutative Yang-Mills action in eq. (2.1) vanish. So for the family of theories studied in this paper, and at first order in θ,

the only classical noncommutative corrections to the ordinary classical action come from the fermionic action in eq. (2.1).

Since we shall formulate the Feynman rules of our theory in terms of ordinary Dirac fermions, we include in it a spectator right-handed fermion, as done in ref. [15]

$$S \to S' = S + \int d^4x\, \bar{\tilde{\psi}}_R i\slashed{\partial}\tilde{\psi}_R, \quad \psi = \begin{bmatrix} \tilde{\psi}_R \\ \psi_L \end{bmatrix}.$$

Again as in ref. [15], we shall regularise the theory by means of dimensional regularisation in $D = 4 + 2\epsilon$ dimensions, using the BMHV scheme for defining γ_5 [16, 17]. In this scheme there is an infinity of dimensionally regularised actions which reduce to (2.1) in the limit $D = 4$, and which differ from one another by evanescent operators [18]. Following [18] we will keep all the vector indices in interaction vertices "four-dimensional", i.e., contracted with the "barred" metric $\bar{g}_{\mu\nu}$; we shall also define the dimensionally regularised $\theta^{\mu\nu}$ as being "four-dimensional". Furthermore, in our computations we shall discard any contribution which has a pole in ϵ but whose numerator is an evanescent operator. Since we shall be dealing with an anomaly free theory, these contributions involving evanescent operators have no physical effects at the one-loop level [18, 19]–although they are needed at two loops and beyond [20] — and are mere artifacts of the regularisation procedure. It is not difficult to convince onself that the famous one-loop $log\,(-Q^2/\mu^2)$ contributions to Green functions are uniquely fixed by the pole contributions to the effective action with no evanescent operators.

Our aim is to compute the one-loop UV divergent part of the effective action involving two fermion fields and no evanescent operator in a manifestly covariant approach, which allows to reconstruct the full contribution to the effective action from a minimum number of diagrams, as was done in ref. [12]. For this we use the background field method [21]. This method amounts to split the gauge field a_μ in a background part b_μ and a quantum part q_μ,

$$a_\mu = b_\mu + q_\mu, \tag{2.3}$$

and choose a gauge fixing which preserves background gauge transformations

$$\delta q_\mu = -i[q_\mu, c],\ \delta b_\mu = D[b]_\mu c,\ D[b]_\mu = \partial_\mu - i[b_\mu,].$$

This gauge fixing is

$$S_{gf} = -\frac{1}{2\alpha} \int d^4x\, (D[b]_\mu q^\mu)^2, \quad S_{gh} = \int d^4x\, \bar{c}D[b]_\mu D[b+q]^\mu c.$$

Adapting to our case the discussions in refs. [21] and [12], introducing the classical fields $\hat{b}_\mu, \hat{\psi}$, the 1PI functional is given by

$$\Gamma[\hat{b}_\mu, \hat{\psi}, \hat{\bar{\psi}}] = \int d^4x \sum_k \sum_n \frac{-i}{(k!)^2} \tilde{\Gamma}^{(n,k)}_{\substack{i_1,..,i_k;\ j_1,..,j_k;\ \mu_1,..,\mu_n \\ a_1,..,a_k}} \prod_{l=1}^k \hat{\bar{\psi}}_{i_l} \prod_{p=1}^k \hat{\psi}_{j_p} \prod_{m=1}^n \hat{b}^{a_m}_{\mu_m}. \tag{2.4}$$

The previous effective action is gauge invariant under gauge transformations of the classical fields $\hat{b}_\mu, \hat{\psi}, \hat{\bar{\psi}}$. The dimensionally regularised version of the effective action above is not
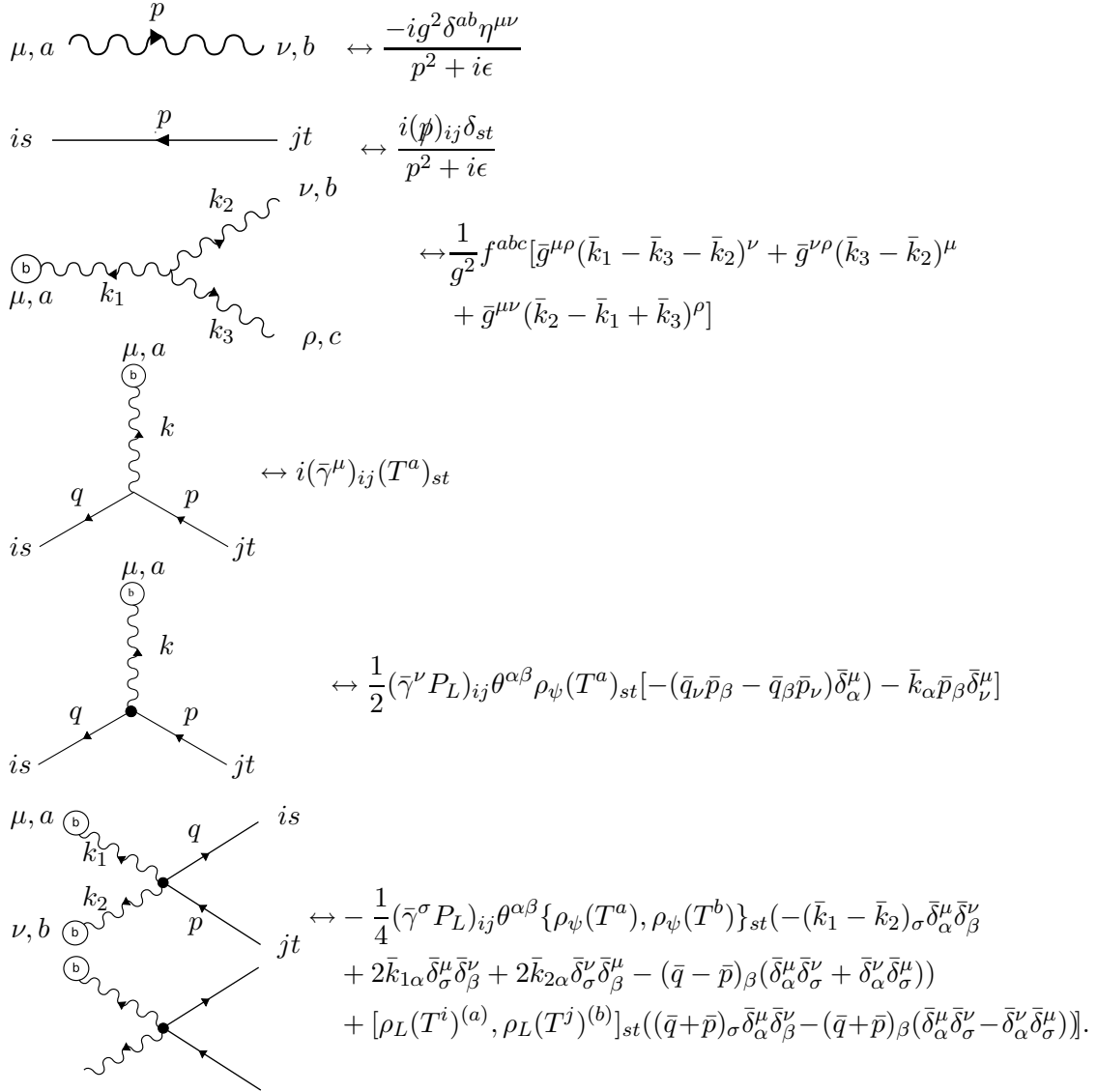
Figure 1. Feynman rules of the noncommutative interactions relevant to our calculations, involving the Dirac fermion ψ.

strictly speaking gauge invariant, i.e, it is gauge invariant modulo an evanescent operator which as we have argued above can be dropped for anomaly- free theories in UV divergent one-loop computations.

Let us notice that $\tilde{\Gamma}^{(n,k)}$ is equivalent to a background 1PI diagram with n background gauge field legs, k fermionic legs and k anti-fermionic legs. (Note that our definitions do not involve any symmetrisation over the background gauge fields). The vertices relevant to our calculations and their associated Feynman rules for $\alpha = 1$ are given in figure 1. In the Feynman rules, the background field legs are denoted with an encircled "b"; the rules are defined without symmetrising over background field legs, in accordance with eq. (2.4).

3 Computation of the UV divergent part of the effective action involving two fermion fields

In this section we shall compute the UV divergent contributions (not involving evanescent operators) to the effective action involving two fermion fields, at one-loop and first order in θ, by calculating the background field 1PI diagrams $\tilde{\Gamma}^{(n,k)}$ with no external quantum field legs of eq. (2.4), using the Feynman rules in figure 1.

To ease the computation we consider the following simplifications, which do not mean a loss of generality of the results:

- Choice of gauge $\alpha = 1$. This choice greatly simplifies the gauge propagator and since the on-shell effective action is independent of the gauge-fixing term –see [22] and references therein–, the conclusions we shall draw from our explicit computations upon taking them on-shell will also be gauge independent.

- Computing a minimum number of diagrams. Since the use of the background field method ensures gauge invariance (modulo one-loop irrelevant evanescent operators) of the result for an anomaly free theory, the full gauge invariant contribution to the UV divergent part with no evanescent operator of the effective action –which is local in the fields– can be reconstructed from a reduced number of 1PI diagrams $\tilde{\Gamma}^{(n,k)}$. These UV divergent contributions to the effective action can be expanded in a basis of independent gauge invariant terms. If their contributions with a given number and types of fields are also independent, then the coefficients in the expansion can be fixed by computing the 1PI diagrams with the same number and types of fields.

In order to identify the diagrams that must be computed, we should start by choosing a basis in 4 dimensions of gauge invariant terms whose integrals are independent. Since it was shown in ref. [15] that noncommutative GUT inspired theories such as the ones under consideration have no four fermion divergences, a little power-counting takes us to the conclusion that we only need to consider terms with two fermion fields. We choose the following ones, for each flavour r:

$$
\begin{aligned}
&s_1^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L f_{\mu\beta} D_\alpha \psi_r, &&s_2^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L f_{\alpha\beta} D_\mu \psi_r, &&s_3^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L \mathcal{D}_\mu f_{\alpha\beta} \psi_r, \\
&s_4^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L f_{\beta\mu} D^\mu \psi_r, &&s_5^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L \mathcal{D}^\mu f_{\beta\mu} \psi_r, &&s_6^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu P_L \mathcal{D}^\nu f_{\mu\nu} \psi_r, \\
&s_7^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu P_L f_{\mu\nu} D^\nu \psi_r, &&s_8^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha{}^{\rho\sigma} P_L \mathcal{D}_\beta f_{\rho\sigma} \psi_r, &&s_9^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha{}^{\rho\sigma} P_L f_{\rho\sigma} D_\beta \psi_r, \\
&s_{10}^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha{}^{\rho\sigma} P_L f_{\beta\sigma} D_\rho \psi_r, &&s_{11}^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha D_\beta D^2 \psi_r, &&s_{12}^r = \theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu D_\mu D^2 \psi_r.
\end{aligned}
\tag{3.1}
$$

In the formulae above, $f_{\mu\nu}$ and $D_\alpha f_{\mu\nu}$ are shorthands for $\rho_r(f_{\mu\nu})$ and $\rho_r(D_\alpha f_{\mu\nu})$. We will omit explicit indications of the representations ρ_r in future formulae; it will be assumed that a Lie-algebra valued field or generator acting on a fermion ψ_r does so in the representation ρ_r. Note that there are other admissible gauge invariant terms, involving symmetric invariant tensors $t^{a_1\cdots a_k}$ of the gauge group, such as $\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu t^{a_1\cdots a_k} T_r^{a_1} \ldots T_r^{a_{k-1}} (f_{\alpha\beta})^{a_k} D_\mu \psi_r$; however, these terms, as will be justified in the next paragraph from the Feynman rules and the inspection of Feynman diagrams, do not appear in the UV divergent part of the
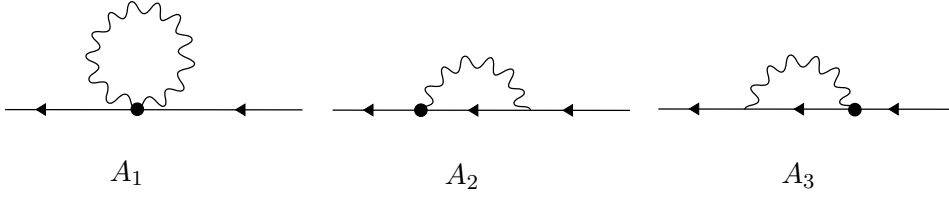
Figure 2. Diagrams contributing to $\tilde{\Gamma}^{(0,1)}$ at order h.

effective action at first order in θ, and may be ignored (also, recall that we are dealing with anomaly safe theories with $t^{abc} = d^{abc} = 0$).

First, let us show that the gauge invariant contributions to the non-evanescent, divergent part of the effective action involving two fermion fields at first order in θ, $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$, can be fixed by only considering Green functions with two fermionic legs and with two fermionic legs and a bosonic leg, respectively. Schematically, the gauge invariant terms that may contribute are of the form $\theta\bar{\psi}D^3\psi$ –whose contributions will appear in Green functions with two fermion fields and between zero and three gauge fields– and $\theta\bar{\psi}(Df)\psi, \theta\bar{\psi}fD\psi$, which will involve contributions to Green functions with two fermion fields and between one and three gauge fields. Given a basis of local gauge invariant terms such that their contributions with a minimum number of fields are independent of each other, then the coefficients of the expansion of $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$ in such basis can be fixed by computing the divergences of the Green functions with the least possible number of fields. This means that for our purposes it suffices to compute the pole part of the Green functions with two fermion fields and two fermion fields and one gauge field, respectively. The diagrams are shown in figures 2 and 3. Using the Feynman rules of figure 1, it is easily seen that the diagrams of figure 2 have a colour structure of the form $T^a_r T^a_r = C_2(r)\mathbb{I}$ –$C_2(r)$ being the second Casimir invariant of the representation r. Their contribution to $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$ is $-i\tilde{\Gamma}^{(0,1)\mathrm{div}}_{ij}\hat{\bar{\psi}}_i\hat{\psi}_j$ (see eq. (2.4)) and given their colour structure it is clear that they fix the coefficients of the terms s_{11}, s_{12} in the expansion of $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$ in the basis of eq. (3.1). Next, the diagrams of figure 3 have colour structures of the form $f^{abc}T^b_r T^c_r = \frac{i}{2}C_2(G)T^a, T^a_r T^b_r T^b_r = C_2(r)T^a_r$, where $C_2(G)$ is the second Casimir in the adjoint representation. These colour structures involve only the generators T^a_r, and therefore it will be possible to express the contribution $-i\tilde{\Gamma}^{(1,1)\mathrm{div}}_{ij(\mu,a)}\hat{\bar{\psi}}_i\hat{\psi}_j\hat{b}^a_\mu$ to $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$ in terms of the three-field contributions of the terms s_1-s_{12} of eq. (3.1); there is no need to consider more complicated colour structures in the basis of gauge invariant terms. Once the terms coming from s_{11}, s_{12} –which, as explained before, are identified from the diagrams of figure 2– are subtracted, one can fix the coefficients of the terms s_1-s_{10} in the expansion of $\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}$ in the basis of eq. (3.1).

The results for the UV divergent contributions of the diagrams of figures 2 and 3 are shown in appendix A. The final result for the gauge invariant $O(\theta)$ contributions to the divergent part of the one-loop effective action is the following,

$$\Gamma^{\mathrm{div},\theta}_{\bar{\psi}\psi}[b_\mu, \psi] = \int d^D x \left\{ \frac{g^2}{192\pi^2\epsilon}\theta^{\alpha\beta}\sum_r C_2(r)\,\bar{\psi}_r\gamma_{\alpha\beta\rho}P_L D^\rho D^2\psi_r + \frac{ig^2}{16\pi^2\epsilon}\sum_r C_2(G) \right.$$
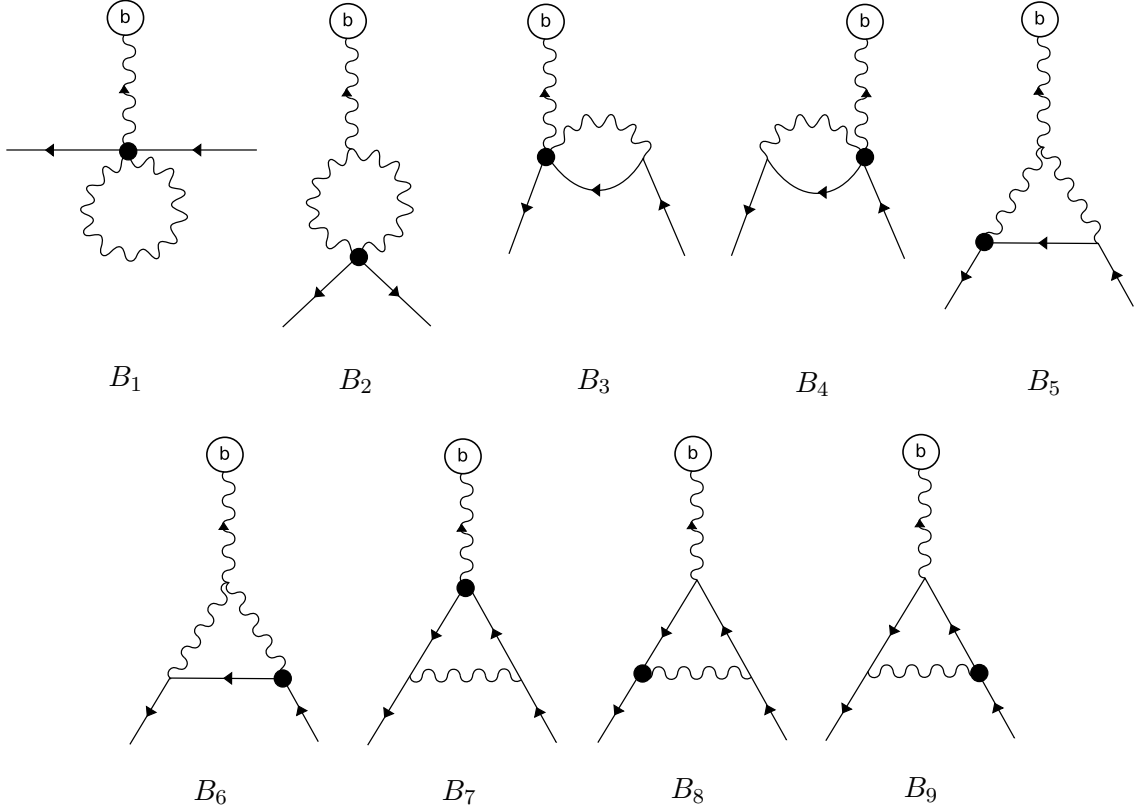
Figure 3. Diagrams contributing to $\tilde{\Gamma}^{(1,1)}$ at order h.

$$\times \left[\frac{1}{6}\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L f_{\mu\beta}D_\alpha\psi_r - \frac{1}{3}\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L f_{\alpha\beta}D_\mu\psi_r \right.$$

$$- \frac{1}{8}\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L \mathcal{D}_\mu f_{\alpha\beta}\psi_r + \frac{5}{6}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L f_{\beta\mu}D^\mu\psi_r$$

$$+ \frac{5}{12}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L \mathcal{D}^\mu f_{\beta\mu}\psi_r - \frac{1}{8}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha{}^{\rho\sigma} P_L \mathcal{D}_\beta f_{\rho\sigma}\psi_r - \frac{1}{16}\theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu P_L \mathcal{D}^\nu f_{\mu\nu}\psi_r \right]$$

$$+ \frac{ig^2}{16\pi^2\epsilon}\sum_r C_2(r)\left[\frac{1}{2}\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L f_{\mu\beta}D_\alpha\psi_r + \frac{1}{8}\theta^{\alpha\beta}\bar{\psi}_r\gamma^\mu P_L \mathcal{D}_\mu f_{\alpha\beta}\psi_r - \frac{3}{2}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L f_{\beta\mu}D^\mu\psi_r \right.$$

$$- \frac{3}{4}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha P_L \mathcal{D}^\mu f_{\beta\mu}\psi_r + \frac{1}{24}\theta^{\alpha\beta}\bar{\psi}_r\gamma_\alpha{}^{\rho\sigma} P_L \mathcal{D}_\beta f_{\rho\sigma}\psi_r + \frac{1}{6}\theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu P_L \mathcal{D}^\nu f_{\mu\nu}\psi_r$$

$$\left.\left. + \frac{1}{12}\theta^{\alpha\beta}\bar{\psi}_r\gamma_{\alpha\beta}{}^\mu P_L f_{\mu\nu}D^\nu\psi_r \right]\right\}, \tag{3.2}$$

where all covariant derivatives and field strengths are evaluated on the background field b_μ, and we have suppressed the hats on the classical fields as well as explicit indications of the representations ρ_r of the field strength and its covariant derivatives to ease the notation. In the formulae above, $C_2(r)$ represents the second Casimir of the representation r, $C_2(G)$ corresponding to the adjoint representation. It is defined as $T_r^a T_r^a = C_2(r)\mathbb{I}_r$. In terms of

the basis s_i of eq. (3.1)

$$\Gamma_{\bar\psi\psi}^{\mathrm{div},\theta}[b_\mu,\psi] = \int d^4x \left(\frac{g^2 i}{16\pi^2\epsilon} \sum_r C_2(G) \left[\frac{1}{6}s_1^r - \frac{1}{3}s_2^r - \frac{1}{8}s_3^r + \frac{5}{6}s_4^r + \frac{5}{12}s_5^r - \frac{1}{16}s_6^r - \frac{1}{8}s_8^r \right] \right.$$

$$\left. + \frac{g^2 i}{16\pi^2\epsilon} \sum_r C_2(r) \left[\frac{1}{2}s_1^r + \frac{1}{8}s_3^r - \frac{3}{2}s_4^r - \frac{3}{4}s_5^r + \frac{1}{6}s_6^r + \frac{1}{12}s_7^r + \frac{1}{24}s_8^r - \frac{i}{12}s_{12}^r \right] \right). \quad (3.3)$$

4 Analysing renormalisability of the matter sector

The objective of this section is to check whether the noncommutative divergences involving two fermion fields of eq. (3.2) can be subtracted by means of multiplicative renormalisation of the coupling constant, the noncommutative parameter θ and fields plus counterterms which vanish on-shell.

First, the the one-loop divergences at order zero in θ are, as is well known, renormalisable by means of multiplicative renormalisations of fields and parameters. These multiplicative renormalisations take the form

$$b_\mu = Z_b^{1/2} b_\mu^R, \qquad \psi = Z_\psi^{1/2} \psi^R, \qquad g = \mu^{-\epsilon} Z_g g^R,,, \qquad \theta_{\mu\nu} = Z_\theta \theta_{\mu\nu}^R, \quad (4.1)$$

with $Z_i = 1 + \delta Z_i$. The parameter μ is the dimensional regularisation scale. Note that in the background field method there is no need to renormalise the quantum field q of eq. (2.3). It is easily seen that gauge invariance forces $\delta Z_b = 0$, while the divergences in the ordinary theory yield

$$\delta Z_\psi^r = \frac{g^2 C_2(r)}{16\pi^2\epsilon}, \quad (4.2)$$

$$\delta Z_g = \frac{g^2}{16\pi^2\epsilon} \left[\frac{11}{6}C_2(G) - \frac{2}{3}\sum_r c_2(r) \right],$$

where $c_2(r)$ is the index of the representation r; its relation with $C_2(r)$ is given in eq. (B.2). Let us introduce the following type of counterterms

$$S^{\mathrm{ct}} = \int d^D x \frac{\delta S}{\delta a_\mu^a(x)} F_\mu^a[a,\psi] + \left(\sum_r \frac{\delta S}{\delta\psi_r(x)} G_r[a,\psi] + \mathrm{c.c.} \right), \quad (4.3)$$

which vanish on-shell due to the equations of motion

$$\frac{\delta S}{\delta a_\mu^a(x)} = \frac{\delta S}{\delta\psi_r(x)} = 0.$$

In order to preserve gauge symmetry, $F^a[a,\psi]$ and $G_r[a,\psi]$ have to transform in 4 dimensions under gauge transformations as follows

$$sF_\mu = -i[F_\mu,\lambda], \quad sG_r = i\lambda G_r.$$

We consider the following F_μ and $G_{r,L}$

$$F_\mu = y_1 \theta^{\alpha\beta} \mathcal{D}_\mu f_{\alpha\beta} + y_2 \theta_\mu{}^\alpha \mathcal{D}^\nu f_{\nu\alpha} + \sum_r y_3^r \theta_\mu{}^\alpha (\bar\psi_r \gamma_\alpha P_L T^a \psi_r) T^a$$

$$+ i\sum_r y_4^r \theta^{\alpha\beta} (\bar\psi_r \gamma_{\mu\alpha\beta} P_L T^a \psi_r) T^a + y_5 \tilde\theta_\mu{}^\beta D^\nu f_{\nu\beta},$$

$$G_{r,L} = k_1^r \theta^{\alpha\beta} f_{\alpha\beta} P_L \psi_r + k_r^2 \theta^{\alpha\beta} \gamma_{\alpha\mu} P_L f_\beta{}^\mu \psi_r + k_3^r \theta^{\alpha\beta} \gamma_{\alpha\mu} P_L D_\beta D^\mu \psi_r + k_4^r \theta^{\alpha\beta} \gamma_{\alpha\beta} P_L D^2 \psi_r$$
$$+ k_5^r \tilde\theta^{\alpha\beta} \gamma_5 P_L f_{\alpha\beta} \psi_r; \quad y_i \in \mathbb{R}, \; k_i \in \mathbb{C}, \tag{4.4}$$

which have the appropriate behaviour under gauge transformations. Note that we only considered a left-handed part for G_r, since all the divergences in eq. (3.2) involve left-handed projectors P_L.

The $O(\theta)$ counterterm action involving two fermion fields obtained by considering the multiplicative renormalisations of eq. (4.1) and the tree-level contributions of the counterterms S^{ct} vanishing on-shell, has the following expansion in the basis of terms s_i:

$$S_{\bar\psi\psi}^{\mathrm{ct},\theta} = ih \int d^D x \sum_{r,i} C_i^r s_i^r, \tag{4.5}$$

$$C_1^r = \frac{1}{2}(\delta Z_\theta + \delta Z_\psi) - (k_2^r)^* - k_2^r, \qquad C_2^r = -\frac{1}{4}(\delta Z_\theta + \delta Z_\psi) + (k_1^r)^* + k_1^r + \frac{i}{2}(k_3^r)^* - \frac{i}{2}k_3^r,$$

$$C_3^r = -iy_1 + k_1^r - \frac{1}{2}k_2^r + \frac{i}{2}(k_3^r)^*, \qquad C_4^r = -(k_2^r)^* - k_2^r - i(k_3^r)^* - ik_3^r - 4i(k_4^r)^*,$$

$$C_5^r = iy_2 - \frac{i}{2g^2}y_3^r - 2i(k_4^r)^* - k_2^r, \qquad C_6^r = -\frac{1}{2g^2}y_4^r - \frac{1}{2}y_5 - i(k_4^r)^* - ik_5^r,$$

$$C_7^r = -2i(k_4^r)^* - i(k_5^r)^* - ik_5^r, \qquad C_8^r = -\frac{1}{2}k_2^r + \frac{i}{2}(k_3^r)^* + \frac{i}{2}k_3^r - ik_5^r,$$

$$C_9^r = \frac{i}{2}(k_3^r)^* + \frac{i}{2}k_3^r - ik_5^r - i(k_5^r)^*, \quad C_{10}^r = (k_2^r)^* - k_2^r + i(k_3^r)^* + ik_3^r,$$

$$C_{11}^r = -(k_3^r)^* - k_3^r - 2(k_4^r)^* - 2k_4^r, \quad C_{12}^r = -(k_4^r)^* + k_4^r.$$

The y_i, k_i of eq. (4.4) also generate, to $O(\theta)$ and at tree-level, terms involving four fermion fields,

$$S_{\bar\psi\psi\bar\psi\psi}^{\mathrm{ct},\theta} = \int d^4 x \sum_{r,s} (y_3^s \theta^{\alpha\beta} + 2y_4^s \tilde\theta^{\alpha\beta})(\bar\psi_r \gamma_\alpha T^a P_L \psi_r)(\bar\psi_s \gamma_\beta T^a P_L \psi_s), \quad \tilde\theta^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\theta_{\mu\nu}. \tag{4.6}$$

As seen in ref. [15], four-fermion divergences are absent in noncommutative GUT compatible theories at one loop, $O(\theta)$. Renormalisability of the divergences involving two and four fermions amounts to demand

$$S_{\bar\psi\psi\bar\psi\psi}^{\mathrm{ct},\theta} = 0, \quad S_{\bar\psi\psi}^{\mathrm{ct},\theta} = -\Gamma_{\bar\psi\psi}^{\mathrm{div},\theta}.$$

where $S_{\bar\psi\psi\bar\psi\psi}^{\mathrm{ct},\theta}$, $S_{\bar\psi\psi}^{\mathrm{ct},\theta}$ and $\Gamma_{\bar\psi\psi}^{\mathrm{div},\theta}$ are given, respectively, by eqs. (4.6), (4.5) and (3.3) The first equation is solved by choosing y_3^r, y_4^r to be flavour independent ($y_i^r = y_i \, \forall \, r$). Solving the second identity in the basis of independent terms s_i^r and projecting the resulting equations into their real and imaginary parts, one gets

$$s_1^r: \quad \frac{1}{2}(\delta Z_\theta + \delta Z_\psi) - 2\mathrm{Re}k_2^r = -\frac{g^2}{16\pi^2\epsilon}\left(\frac{1}{6}C_2(G) + \frac{1}{2}C_2(r)\right),$$

$$s_2^r: \quad -\frac{1}{4}(\delta Z_\theta + \delta Z_\psi) + 2\mathrm{Re}k_1^r + \mathrm{Im}k_3^r = \frac{g^2 C_2(G)}{48\pi^2\epsilon},$$

$$s_3^r: \quad y_1 - \frac{1}{2}\mathrm{Re}k_3^r - \mathrm{Im}k_1^r + \frac{1}{2}\mathrm{Im}k_2^r = 0, \quad \frac{1}{2}\mathrm{Im}k_3^r + \mathrm{Re}k_1^r - \frac{1}{2}\mathrm{Re}k_2^r = \frac{g^2(C_2(G) - C_2(r))}{128\pi^2\epsilon},$$

$$s_4^r: \quad \mathrm{Re}k_3^r + 2\mathrm{Re}k_4^r = 0, \quad -2\mathrm{Re}k_2^r - 4\mathrm{Im}k_4^r = -\frac{g^2}{16\pi^2\epsilon}\left(\frac{5}{6}C_2(G) - \frac{3}{2}C_2(r)\right),$$

$$s_5^r: \quad \frac{1}{2g^2}y_3^r - y_2 + 2\mathrm{Re}k_4^r + \mathrm{Im}k_2^r = 0, \quad -\mathrm{Re}k_2^r - 2\mathrm{Im}k_4^r = -\frac{g^2}{16\pi^2\epsilon}\left(\frac{5}{12}C_2(G) - \frac{3}{4}C_2(r)\right),$$

$$s_6^r: \quad \mathrm{Re}k_4^r + \mathrm{Re}k_5^r = 0, \quad -\frac{1}{2g^2}y_4^r - \frac{1}{2}y_5 - \mathrm{Im}k_4^r + \mathrm{Im}k_5^r = -\frac{g^2}{16\pi^2\epsilon}\left(-\frac{1}{16}C_2(G) + \frac{1}{6}C_2(r)\right),$$

$$s_7^r: \quad \mathrm{Re}k_4^r + \mathrm{Re}k_5^r = 0, \quad -2\mathrm{Im}k_4^r = -\frac{g^2C_2(r)}{192\pi^2\epsilon},$$

$$s_8^r: \quad -\mathrm{Re}k_3^r + \mathrm{Re}k_5^r + \frac{1}{2}\mathrm{Im}k_2^r = 0, \quad \mathrm{Im}k_5^r - \frac{1}{2}\mathrm{Re}k_2^r = -\frac{g^2}{16\pi^2\epsilon}\left(-\frac{1}{8}C_2(G) + \frac{1}{24}C_2(r)\right),$$

$$s_9^r: \quad -\mathrm{Re}k_3^r + 2\mathrm{Re}k_5^r = 0,$$

$$s_{10}^r: \quad \mathrm{Im}k_2^r - \mathrm{Re}k_3^r = 0,$$

$$s_{11}^r: \quad -\mathrm{Re}k_3^r - 2\mathrm{Re}k_4^r = 0,$$

$$s_{12}^r: \quad -2\mathrm{Im}k_4^r = -\frac{g^2C_2(r)}{192\pi^s\epsilon}.$$

The equations are compatible, and we find the following family of solutions

$$y_1 = \mathrm{Im}k_1^r, \qquad\qquad\qquad y_3^r = 2g^2y_2,$$

$$y_4^r = -y_5g^2 - \frac{g^4}{384\pi^2}(16C_2(r) - 13C_2(G)), \qquad Z_\theta = -Z_\psi - \frac{g^2}{48\pi^2\epsilon}(13C_2(r) - 4C_2(G)),$$

$$\mathrm{Re}k_1^r = -\frac{1}{2}\mathrm{Im}k_3^r - \frac{g^2}{384\pi^2\epsilon}(13C_2(r) - 8C_2(G)), \quad \mathrm{Im}k_5^r = -\frac{g^2}{384\pi^2\epsilon}(11C_2(r) - 8C_2(G)),$$

$$\mathrm{Im}k_4^r = \frac{g^2C_2(r)}{384\pi^2\epsilon}, \qquad\qquad\qquad \mathrm{Re}k_2^r = -\frac{5g^2}{192\pi^2\epsilon}(2C_2(r) - C_2(G)),$$

$$\mathrm{Im}k_2^r = \mathrm{Re}k_3^r = 2\mathrm{Re}k_5^r = -2\mathrm{Re}k_4^r. \tag{4.7}$$

First, note that y_1, y_2, y_5 and δZ_θ must be flavour independent (see eq. (4.4)), and so must be y_3, y_4 for the cancellation of the four fermion divergences to be preserved under the renormalisation procedure, as was previously seen. Looking at the solutions in eq. (4.7) and imposing flavour independence, it is clear that one must require that all flavours have identical $C_2(r)$; this can be achieved by considering all fields in the same representation or also in its conjugate.

Since the counterterms of eqs. (4.3) and (4.4), which are dependent on the parameters y_i, and k_i, vanish on-shell, we have that the corresponding divergences in the effective action are physically irrelevant since they will cancel out when computing the S matrix. Looking at the expansion in the counterterm action of eq. (4.5), it is clear that the only divergences surviving on-shell are those associated with the multiplicative renormalisation of the gauge coupling constant and noncommutativity parameter θ.

5 No $O(\theta)$ UV divergent contributions in the gauge sector

The results of the previous section, together with those of ref. [15], show that the matter sector of the one-loop, order θ effective action is renormalisable. It remains to see if the

gauge sector is also renormalisable. In all cases analysed in the literature so far, the gauge sector of noncommutative gauge theories in the enveloping algebra approach turned out to be one-loop renormalisable at order θ. We show in this section that, at one-loop, there are no UV divergent contributions of first order in θ to the part of the background-field effective action, $\Gamma[\hat{b}_\mu, \hat{\psi} = 0, \hat{\bar{\psi}} = 0]$ which only depends on the gauge field.

Possible UV divergences in the gauge sector can be of two types, depending on whether they involve ϵ tensors or not. Since any vector-like contribution to the effective action can always be regularised in a gauge-invariant way in the framework of dimensional regularisation, the allowed $O(\theta)$ vector-like UV divergences can only be a combination of the terms $\mathrm{Tr}\,\theta^{\alpha\beta} f_{\alpha\beta} f_{\mu\nu} f^{\mu\nu}$ and $\mathrm{Tr}\,\theta^{\alpha\beta} f_{\alpha\mu} f_{\beta\nu} f^{\mu\nu}$, which vanish for anomaly safe groups since they involve vanishing symmetrised $\mathrm{Tr}\,T^a\{T^b, T^c\}$. It only remains to show that there are no UV divergences involving ϵ tensors. These divergences would come from fermionic loops, since the ϵ tensors arise from traces of γ matrices. The one-loop fermionic contributions to the gauge sector of the effective action can be computed in a clever way, to all orders in θ, using the technique used in refs. [2] and [13]. By defining appropriately the dimensionally regularised interactions –recall that there is an infinity of choices, differing by evanescent contributions– a change of variables can be done in the fermionic path integral which amounts to inverting the SW map and whose Jacobian is unity. The diagrams to compute involve vertices with noncommutative fermions, in which the noncommutative phase factors are independent of the loop momenta. Then, as done in ref. [2], it can be easily seen that these diagrams have vanishing UV divergent contributions involving ϵ tensors.

We have seen that there are no UV divergences contributions in the gauge sector at one-loop and first order in θ. Now, since at tree-level there are no $O(\theta)$ contributions only involving gauge fields, there is no conflict with the multiplicative renormalisations of eq. (4.2), and thus the gauge sector is one-loop renormalisable up to first order in θ.

6 Summary, conclusions and outlook

In this paper we have computed the UV divergent contributions, involving two fermions and an arbitrary number of gauge fields, to the background field effective action of noncommutative, anomaly safe GUT inspired theories with no scalars. We have done the computation at one-loop and first order in the noncommutativity parameter θ. We have shown that those UV divergences can be renormalised by means of the ordinary multiplicative renormalisations of the coupling constant and fields, along with a multiplicative renormalisation of the noncommutative parameter θ and with the introduction of θ-dependent counterterms which vanish on-shell, provided the irreps furnished by the matter fermionic fields share the same second-degree Casimir invariant. It is obvious that this condition on the second-degree Casimir invariant is automatically fulfilled by the fermionic matter content of the phenomenologically relevant ordinary $SO(10)$ and E_6 GUTs. We have also shown that the gauge sector of these theories receives no linear one-loop UV divergent radiative corrections which are of first order in θ.

Our results, together with those of ref. [15] proving the absence of 4 fermion UV divergences in the one-loop effective action of noncommutative GUT inspired theories at

first order in θ, show that the theories considered in this paper are renormalisable on-shell at one-loop and first order in θ. We have thus seen that, at one-loop and first order in θ, only the renormalisation of the coupling constant and the noncommutative parameter θ –the two free physical parameters of the classical theory– are needed to obtain renormalised S matrix elements. This is the first time in which a noncommutative gauge theory defined by means of Seiberg-Witten map, with fermions in representations other than the adjoint, has been shown to have this property. This result clearly favours the consideration of GUT compatible noncommutative theories over their nonrenormalisable brethren. The only other known examples of one-loop, $O(\theta)$ renormalisable noncommutative gauge theories involve $SU(N)$ adjoint Majorana fermions in a supersymmetric setting [12].

A pressing open problem is the study, at one-loop and first order in θ, of the renormalisability of the noncommutative GUT theories obtained by adding a noncommutative Higgs and Yukawa sectors –through the hybrid Seiberg-Witten map of ref. [4]– to the noncommutative theories considered here. The computations involved in this study are far more lengthy that the already long calculations carried out in this paper and will certainly deserve to be the content of a different paper. We hope that the results presented here will encourage people to further analyse the properties of noncommutative GUT theories.

Acknowledgments

A   Divergent contributions to the diagrams of figures 2 and 3

Here we give the pole part of the Feynman diagrams depicted in figures 2 and 3, computed in dimensional regularisation with $D = 4 + 2\epsilon$ dimensions.

$$A_1 = 0,$$

$$A_2 = -\frac{g^2}{96\pi^2\epsilon}\bigoplus_r C_2(r)\mathbb{I}_r\theta^{\alpha\beta}\gamma_\beta P_L p^2 p_\alpha - \frac{g^2}{384\pi^2\epsilon}\bigoplus_r C_2(r)\mathbb{I}_r\theta^{\alpha\beta}\gamma_{\rho\alpha\beta}P_L p^2 p^\rho,$$

$$A_3 = \frac{g^2}{96\pi^2\epsilon}\bigoplus_r C_2(r)\mathbb{I}_r\theta^{\alpha\beta}\gamma_\beta P_L p^2 p_\alpha - \frac{g^2}{384\pi^2\epsilon}\bigoplus_r C_2(r)\mathbb{I}_r\theta^{\alpha\beta}\gamma_{\rho\alpha\beta}P_L p^2 p^\rho,$$

$$B_1 = 0,$$

$$B_2 = -\frac{g^2}{8\pi^2\epsilon}\bigoplus_r C_2(G)T_r^A(\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta} - \not{p}k_\alpha\theta^{\alpha\mu} + \not{k}p_\alpha\theta^{\mu\nu})P_L,$$

$$B_3 = -\frac{g^2}{8\pi^2\epsilon}\bigoplus_r C_2(r)T_r^A\Big(\frac{1}{8}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta} + \frac{1}{8}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta} + \frac{1}{24}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta} + \frac{1}{8}\not{k}p_\alpha\theta^{\alpha\mu} + \frac{1}{24}\not{p}p_\alpha\theta^{\alpha\mu}$$

$$-\frac{1}{8}k\cdot p\gamma_\beta\theta^{\mu\beta} - \frac{1}{8}p^2\gamma_\beta\theta^{\mu\beta} + \frac{1}{8}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta} + \frac{1}{24}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta} + \frac{1}{48}p^2\gamma^\mu{}_{\alpha\beta}\theta^{\alpha\beta}$$

$$-\frac{1}{8}k^\rho p^\sigma \gamma_{\rho\sigma\beta}\theta^{\mu\beta}\Big)P_L$$

$$-\frac{g^2}{32\pi^2\epsilon}\bigoplus_r C_2(G)T_r^A\left(-\frac{1}{4}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{4}\slashed{p}p_\alpha\theta^{\alpha\mu}+\frac{1}{4}p^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{4}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}\right)P_L,$$

$$B_4=-\frac{g^2}{8\pi^2\epsilon}\bigoplus_F C_2(F)T_F^A\Big(\frac{1}{8}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}-\frac{1}{6}k^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{6}\slashed{k}k_\alpha\theta^{\alpha\mu}+\frac{1}{24}\slashed{p}k_\alpha\theta^{\alpha\mu}$$

$$+\frac{1}{24}k^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{24}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}\slashed{k}p_\alpha\theta^{\alpha\mu}-\frac{1}{24}\slashed{p}p_\alpha\theta^{\alpha\mu}+\frac{1}{4}k^2\gamma_\beta\theta^{\mu\beta}-\frac{3}{8}k\cdot p\gamma_\beta\theta^{\mu\beta}$$

$$+\frac{1}{8}p^2\gamma_\beta\theta^{\mu\beta}+\frac{1}{6}k^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{6}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{24}k^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}+\frac{1}{24}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}$$

$$+\frac{1}{48}k^2\gamma^\mu{}_{\alpha\beta}\theta^{\alpha\beta}-\frac{1}{24}k\cdot p\gamma^\mu{}_{\alpha\beta}\theta^{\alpha\beta}+\frac{1}{48}p^2\gamma^\mu{}_{\alpha\beta}\theta^{\alpha\beta}+\frac{1}{8}k^\rho p^\sigma\gamma_{\rho\sigma\beta}\theta^{\mu\beta}\Big)P_L$$

$$-\frac{g^2}{32\pi^2\epsilon}\bigoplus_F C_2(G)T_F^A\left(\frac{1}{4}k^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{4}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{4}\slashed{k}k_\alpha\theta^{\alpha\mu}-\frac{1}{4}\slashed{p}k_\alpha\theta^{\alpha\mu}\right.$$

$$-\frac{1}{4}k^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{4}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{4}\slashed{k}p_\alpha\theta^{\alpha\mu}+\frac{1}{4}\slashed{p}p_\alpha\theta^{\alpha\mu}-\frac{1}{4}k^2\gamma_\beta\theta^{\mu\beta}+\frac{1}{2}k\cdot p\gamma_\beta\theta^{\mu\beta}$$

$$\left.-\frac{1}{4}p^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{4}k^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}+\frac{1}{4}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}+\frac{1}{4}k^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{4}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}\right)P_L,$$

$$B_5=\frac{g^2}{32\pi^2\epsilon}\bigoplus_r C_2(G)T_r^A\left(\frac{5}{6}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}-\frac{1}{6}k^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{3}\slashed{k}k_\alpha\theta^{\alpha\mu}-\frac{1}{3}\slashed{p}k_\alpha\theta^{\alpha\mu}\right.$$

$$-\frac{1}{6}k^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{5}{6}\slashed{k}p_\alpha\theta^{\alpha\mu}-\frac{1}{6}\slashed{p}p_\alpha\theta^{\alpha\mu}+\frac{1}{3}k^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{2}k\cdot p\gamma_\beta\theta^{\mu\beta}$$

$$+\frac{1}{6}p^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{3}k^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{6}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}+\frac{2}{3}k^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{6}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}$$

$$-\frac{1}{24}k^\mu k^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}+\frac{1}{12}p^\mu k^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}+\frac{1}{24}k^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}-\frac{1}{12}p^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}$$

$$\left.+\frac{1}{6}k^\rho p^\sigma\gamma_{\rho\sigma\beta}\theta^{\mu\beta}\right)P_L,$$

$$B_6=\frac{g^2}{32\pi^2\epsilon}\bigoplus_r C_2(G)T_r^A\Big(\frac{5}{6}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}+\frac{1}{3}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{2}\slashed{p}k_\alpha\theta^{\alpha\mu}-\frac{1}{6}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{2}{3}\slashed{k}p_\alpha\theta^{\alpha\mu}$$

$$+\frac{1}{6}\slashed{p}p_\alpha\theta^{\alpha\mu}-\frac{1}{6}k\cdot p\gamma_\beta\theta^{\mu\beta}-\frac{1}{6}p^2\gamma_\beta\theta^{\mu\beta}+\frac{1}{3}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{2}k^\rho p_\alpha\gamma^\mu_{\rho\beta}\theta^{\alpha\beta}-\frac{1}{6}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}$$

$$+\frac{1}{24}k^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}-\frac{1}{12}p^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}-\frac{1}{6}k^\rho p^\sigma\gamma_{\rho\sigma\beta}\theta^{\mu\beta}\Big)P_L,$$

$$B_7=-\frac{g^2}{16\pi^2\epsilon}\bigoplus_r\left(C_2(r)-\frac{1}{2}C_2(G)\right)T_r^A\left(-\frac{1}{3}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}-\frac{1}{12}k^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}\right.$$

$$\left.+\frac{1}{4}\slashed{k}k_\alpha\theta^{\alpha\mu}-\frac{1}{6}\slashed{p}k_\alpha\theta^{\alpha\mu}-\frac{1}{3}\slashed{k}p_\alpha\theta^{\alpha\mu}+\frac{1}{12}k^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{6}k\cdot p\gamma_\beta\theta^{\mu\beta}-\frac{1}{4}k^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}\right)P_L,$$

$$B_8=-\frac{g^2}{16\pi^2\epsilon}\bigoplus_F\left(C_2(F)-\frac{1}{2}C_2(G)\right)T_F^A\left(\frac{1}{6}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}-\frac{1}{3}k^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{7}{12}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}\right.$$

$$-\frac{1}{6}\slashed{k}k_\alpha\theta^{\alpha\mu}+\frac{1}{12}\slashed{p}k_\alpha\theta^{\alpha\mu}+\frac{1}{6}k^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{5}{12}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{6}\slashed{k}p_\alpha\theta^{\alpha\mu}-\frac{1}{12}\slashed{p}p_\alpha\theta^{\alpha\mu}$$

$$+\frac{1}{6}k^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{4}k\cdot p\gamma_\beta\theta^{\mu\beta}+\frac{1}{12}p^2\gamma_\beta\theta^{\mu\beta}-\frac{1}{6}k^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{12}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}$$
$$+\frac{1}{3}k^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{12}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{12}k^\mu k^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}+\frac{1}{24}p^\mu k^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}-\frac{1}{24}k^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}$$
$$+\frac{1}{12}p^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}+\frac{1}{8}k^2\gamma_{\alpha\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{8}k\cdot p\gamma_{\alpha\beta}{}^\mu\theta^{\alpha\beta}+\frac{1}{12}k^\rho p^\sigma\gamma_{\rho\sigma\beta}\theta^{\mu\beta}\bigg)P_L,$$

$$B_9=-\frac{g^2}{16\pi^2\epsilon}\bigoplus_F\bigg(C_2(F)-\frac{1}{2}C_2(G)\bigg)T_F^A\bigg(\frac{1}{6}\gamma^\mu k_\alpha p_\beta\theta^{\alpha\beta}+\frac{1}{6}p^\mu k_\alpha\gamma_\beta\theta^{\alpha\beta}-\frac{1}{4}k^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}$$
$$+\frac{5}{12}p^\mu p_\alpha\gamma_\beta\theta^{\alpha\beta}+\frac{1}{12}\not{k}p_\alpha\theta^{\alpha\mu}+\frac{1}{12}\not{p}p_\alpha\theta^{\alpha\mu}-\frac{1}{12}k\cdot p\gamma_\beta\theta^{\mu\beta}-\frac{1}{12}p^2\gamma_\beta\theta^{\mu\beta}+\frac{1}{6}p^\rho k_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}$$
$$-\frac{1}{4}k^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{12}p^\rho p_\alpha\gamma_{\rho\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{8}p^\mu k^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}-\frac{1}{24}k^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}+\frac{1}{12}p^\mu p^\rho\gamma_{\rho\alpha\beta}\theta^{\alpha\beta}$$
$$+\frac{1}{8}k\cdot p\gamma_{\alpha\beta}{}^\mu\theta^{\alpha\beta}-\frac{1}{12}k^\rho p^\sigma\gamma_{\rho\sigma\beta}\theta^{\mu\beta}\bigg)P_L,$$

In the formulae above, $\mathbb{I}_r$ denotes the identity operator in the linear space defined by the representation r.

B Beta functions of the physical couplings g and θ

It is an elementary exercise to work out the one-loop beta functions, β_g and β_θ, of g and θ, respectively, which are the only physical couplings of the theory. One gets the following results

$$\beta_g=-\frac{g^3}{16\pi^2}\bigg(\frac{11}{3}c_2(G)-\frac{4}{3}\sum_r c_2(r)\bigg),$$
$$\beta_\theta=-\frac{g^2\theta}{6\pi^2}(C_2(G)-4C_2(r)). \tag{B.1}$$

$C_2(r)$ represents the second-degree Casimir invariant of the representation r, while $c_2(r)$ is the index of the representation. Both are related by the relation

$$C_2(r)=c_2(r)\frac{N(G)}{N(r)}, \tag{B.2}$$

$N(r)$ being the dimension of the representation r, and G denoting the adjoint representation. The β function for the gauge coupling g is the same as in the commutative theory. The β function for θ, due to the presence of matter, has generically the opposite sign as that of the beta function for the noncommutative parameter that was computed for noncommutative pure gauge theories in ref. [9]. β_θ in eq. (B.1) can be seen to be positive for E_6 and SO(10) representations with dimensions less than 100000 and 12000, respectively, using the data in ref. [23].

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